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COMPARISON OF THREE ANALYTIC METHODS (PSM, ADM, HAM) IN SOLVING A LINEAR AND NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper a linear and a non-linear fractional differential equations are considered to solve the two differential equations analytically by Power Series method (PSM), Adomian Decomposition Method (ADM) and Homotopy Analysis Method (HAM). Solutions obtained by above three methods are identically equal calculated upto the terms containing $X^{9/2}$

Key Words: Fractional linear and non-linear Differential Equation; Power Series method; Adomain Decomposition Method; Homotopy Analysis Method; Caputo Derivative.

MSC No.(2010): 37 L 65, 90 C 59

1. Introduction. Fractional differential equations are generalization of classical differential equations of integer order and have been used successfully to modify many existing models of physical process for improving results. Fractional calculus is a field of mathematical study that grows out of the traditional definition of calculus, both integral and differential operators in much the same way as fractional exponent is an outgrowth of the exponents with integer value. Apart from diverse areas of mathematics, fractional differential equations arise in rheology (Trmohoy, 2003), visco-elasticity (Soczkiewicz, 2002), Ultrasonic Wave propagation in human cancellous bone (Sebaa, Fellah, Lauriks and Depollier, 2006), Modelling of speech signals (Assaleh and Ahmad, 2007), Modelling of the Cardiac Tissue (Magin, 2008), Sound waves propagation in Rigid Porous Materials (Fellah and Depollier, 2002), Lateral and Longitudinal control of autonomous Vehicles (Suarez, Vinagre, Calderon, Monje and Chen, 2004), Fluid Mechanics (Kulish and Lage, 2002), Edge detection (Mathieu, Melchior, Oustaloup and Ceyral, 2003) and in many other areas. There is no doubt that the fractional differential equation has become an exciting new mathematical model for representing many diverse problems in mathematics, science and engineering. But the solutions of these equations involve rigorous mathematical techniques.

The concept of fractional calculus (fractional derivatives and fractional integral) is not new. In 1695 L'Hospital asked Leibnitz, one of the inventors of calculus the question as to the meaning of $\frac{d^n y}{dx^n}$, if $n = 1/2$ i.e., "if n is fraction"?

Leibnitz replied that "$d^{1/2}x$ will be equal to $x\sqrt{dx : x}$". After that discussion on this idea was untouched till the beginning of 20th Century.

The whole theory of fractional derivatives and integrals was established in the 2nd half of the 19th Century. Caputo and Mainnardi (Caputo and Mainnardi, 1971 and Caputo, 1966) and Caputo (1974) found good agreement with experimental results when using fractional derivatives for description of viscoelastic materials and established the connection between fractional derivatives and the theory of linear viscoelasticity. Adolfsson et al (Mainardi, 2010) constructed a new fractional order model of viscoelasticity.

A lot of works have been done by using fractional derivatives for a better description of considered material properties. Several numerical methods for solving fractional differential equations have been evolved. A new analytical approach that can be applied to solve nonlinear fractional differential equations (NFDE) is to employ the Homotopy Analysis Method (HAM) (Golmankhaneh, Golmankhaneh and Baleanu, 2011 and Liao, 1995, 1997, 2003). The HAM solution for Nonlinear Fractional Differential Equation (NFDE) were presented by D. Das (Das, Ray, Bera and Sarkar, 2015). Adomain Decomposition Method (Adomian, 1988, 1994) provided analytical solutions as well as numerical approximate solutions to both linear and non-linear differential equations without linearization or discretization. The ADM is also implemented to give analytical solutions for the class of Lane-Emden Equation (Das, Ghosh, Datta, Bera, Sahu and Ray, 2015). These methods are different from that of perturbation method.

Recent seminars, symposia and proceedings have highlighted the application of fractional calculus in physics, continuum mechanics etc.

So, it has become necessary to solve fractional differential equations. Several methods are available to solve fractional differential equations, for example PSM, ADM, HAM, ete. In this paper we solve a linear and a non-linear fractional differential equation and have shown that the solutions obtained by the above methods are found to be exactly same calculating up to nine terms. If we proceed further, it can be shown that the solutions by the above methods will be the same up to the desired terms.

2. Formulation of the Problem, for linear fractional differential equation

2.1 Mathematical preliminaries on Fractional Calculus. To solve the problems of fractional differential equations many definitions of fractional calculus

have been developed. The most frequently encountered definitions include Riemann-Liouville, Caputo, Wely. Rize fractional operator. We introduce the following definitions [23,24] in the present analysis.

2.1.1 Definition. Let $\alpha \in \mathbb{R}^+$. The integral operator I^α defined on the usual Lebesgue space $L(a,b)$ by

$$I^\alpha f(x) = \frac{d^{-\alpha} f(x)}{dx^{-\alpha}} = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)$$
$$I^0 f(x) = f(x)$$

for $x \in [a,b]$ is called Riemann-Liouville fractional integral operator of order α (≥ 0). It has the following properties:

(i) $I^\alpha f(x)$ exists for any $x \in [a,b]$.

(ii) $I^\alpha I^\beta f(x) = I^{\alpha+\beta} f(x)$

(iii) $I^\alpha I^\beta f(x) = I^\beta I^\alpha f(x)$

(iv) $I^\alpha x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$ where $f(x) \in L[a,b]$, $\alpha, \beta \geq 0$, $\gamma > -1$.

2.1.2 Definition. The Riemann-Liouville definition of fraction order derivative is

$${}^{RL}_{0}D^\alpha_x f(x) = \frac{d^n}{dx^n}\, {}_0I^{n-\alpha}_x f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} f(t)\, dt$$

where n is an integer that satisfies $\alpha \in (n-1, n)$.

2.1.3 Definition. A modified fractional differential operator ${}^{c}_{0}D^\alpha_x$ proposed by Caputo is given by

$${}^{c}_{0}D^\alpha_x f(x) = {}_0I^{n-\alpha}_x \frac{d^n}{dx^n} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f^{(n)}(t) dt,$$

where $\alpha \in \mathbb{R}^+$ is the order of operation and n is an integer that satisfies $a \in (n-1, n)$.

It has the following two basic properties [25]:

(i) If $f \in L_\infty(a,b)$ or $f \in C[a,b]$ and $\alpha > 0$ then ${}^{c}_{0}D^\alpha_x\, {}_0I^\alpha_x f(x) = f(x)$

(ii) If $f \in C^n[a,b]$ and if $\alpha > 0$ then

$$ {}_0I_x^{\alpha}\, {}_0^cD_x^{\alpha} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0+)}{k!} x^k; \quad \alpha \in (n-1, n). $$

2.1.4 Definition. For m being the smallest integer that exceeds α, the Caputo time fractional derivative operator of order $\alpha > 0$, is defined as in [26]

$$ D_t^{\alpha} u(x,t) = \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \begin{cases} \dfrac{\partial^m u(x,\xi)}{\partial \xi^m} & ; \quad \alpha = m \in \mathbb{N} \\ \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t (t-\xi)^{m-\alpha-1} \frac{\partial^m u(x,\xi)}{\partial \xi^m} d\xi & ; \quad m-1 \le \alpha < m \end{cases} $$

2.1.5 Relation between Caputo derivative and Riemann Liouville derivative :

$$ {}_0^cD_t^{\alpha} f(x) = {}_0^{RL}D_t^{\alpha} f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0+)}{\Gamma(k-\alpha+1)} x^{k-\alpha}; \;\; \alpha \in (m-1, m) $$

Integrating by parts, we get the following formulae as given in [27]:

(i) $$ \int_a^b g(x)\, {}_a^cD_x^{\alpha} f(x)dx = \int_a^b f(x)\, {}_x^{RL}D_b^{\alpha} g(x)dx + \sum_{j=0}^{n-1} \left[{}_x^{RL}D_b^{\alpha+j-n} g(x)\, {}_x^{RL}D_b^{n-1-j} f(x) \right]_a^b $$

(ii) For $n = 1$, $$ \int_a^b g(x)\, {}_a^cD_x^{\alpha} f(x)dx = \int_a^b f(x)\, {}_x^{RL}D_b^{\alpha} g(x)dx + \left[{}_xI_b^{1-\alpha} g(x) f(x) \right]_a^b $$

2.2 Linear fractional differential equation. We consider the linear fractional differential equation

$$ \frac{dy}{dx} + \frac{{}^cd^{\alpha} y(x)}{dx^{\alpha}} + y(x) = x \tag{2.1} $$

where α, $0 < \alpha < 1$ is a rational number and initial condition is $y(0) = 1$.

Here $\dfrac{{}^cd^{\alpha} y(x)}{dx^{\alpha}} \equiv {}^cD^{\alpha} y(x)$ is the Caputo derivative of $y(x)$ of order 'α'.

We define $D \equiv \dfrac{d}{dx}$ and ${}^cD^{\alpha} \equiv \dfrac{{}^cd^{\alpha}}{dx^{\alpha}}$, then the differential equation (2.1) takes the form

$$ Dy(x) + {}^cD^{\alpha} y(x) + y(x) = x \tag{2.2} $$

We consider $0 < \alpha < p/k < 1$, where p and k are integers which are prime to each other.

Then $p < k \Rightarrow k - p > 0$.

3. Solution of the problem by different methods

(a) Power Series Method :

Applying I^α on both sides of equation (2.2), we get

$$I^\alpha Dy(x) + I^\alpha\, {}^cD^\alpha y(x) + I^\alpha y(x) = I^\alpha(x) \tag{3.1}$$

The integral operator I^α according to the Riemann-Liouville fractional integral operator of order 'α' is :

$$I^\alpha x^\gamma = \frac{\lceil \gamma+1}{\lceil \alpha+\gamma+1} x^{\gamma+\alpha} \tag{3.2}$$

and $I^\alpha\, {}^cD^\alpha y(x) = y(x) - \sum_{q=0}^{m-1} \frac{y^q(0)}{q!} x^q$, where $m-1 < \alpha < m$. In this paper, we have $0 < \alpha < 1$, i.e., $m = 1$

$$\therefore \quad I^\alpha\, {}^cD^\alpha y(x) = y(x) - y(0) \tag{3.3}$$

Let

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n/k}, \quad \text{where } 0 < \alpha < \frac{p}{k} < 1 \tag{3.4}$$

Then $$Dy(x) = \sum_{n=1}^{\infty} \left(\frac{n}{k}\right) a_n x^{\frac{n}{k}-1}$$

$$\therefore \quad I^\alpha Dy(x) = \sum_{n=1}^{\infty} \left(\frac{n}{k}\right) a_n \frac{\lceil \frac{n}{k}\, x^{\frac{n}{k}-1+\frac{p}{k}}}{\lceil \frac{n}{k}-1+\frac{p}{k}+1} = \sum_{n=1}^{\infty} \left(\frac{n}{k}\right) a_n \frac{\lceil \frac{n}{k}}{\lceil \frac{n}{k}+\frac{p}{k}} x^{\frac{n}{k}-1+\frac{p}{k}}$$

$$= \sum_{n=1}^{\infty} a_n \frac{\lceil \frac{n}{k}+1}{\lceil \frac{n+p}{k}} x^{\frac{n-k+p}{k}} \tag{3.5}$$

$$\text{and } I^\alpha\, {}^cD^\alpha y(x) = y(x) - y(0) = \sum_{n=1}^{\infty} a_n x^{n/k} \tag{3.6}$$

with $a_0 = 1$, from (3.4).

Again

$$I^\alpha y(x) = \sum_{n=0}^{\infty} a_n \frac{\lceil \frac{n+k}{k}}{\lceil \frac{n+k+p}{k}} x^{\frac{n+p}{k}} \tag{3.7}$$

Substitute the values of $I^{\alpha}Dy(x)$, $I^{\alpha}\,{}^{c}D^{\alpha}y(x)$ and $I^{\alpha}y(x)$ from (3.5), (3.6) and (3.7) respectively to the differential equation (3.1) we get

$$\sum_{n=1}^{\infty}\frac{\lceil(n+k)/k}{\lceil(n+p)/k}a_n x^{(n-(k-p))/k}+\sum_{n=1}^{\infty}a_n x^{n/k} +\sum_{n=0}^{\infty}a_n\frac{\lceil(n+k)/k}{\lceil(n+p+k)/k}x^{(n+p)/k}=\frac{x^{1+\alpha}}{\lceil 2+\alpha} \tag{3.8}$$

From the last term of L.H.S. of (3.8),

$$a_n=0 \ \text{ for } \ n=1,2,\cdots,k-p. \tag{3.9}$$

Putting $j=n+p$ in the third term and rewriting (3.8), we get

$$\sum_{n=k-p+1}^{\infty}a_n\frac{\lceil(n+k)/k}{\lceil(n+p)/k}x^{(n-(k-p))/k}+\sum_{j=1}^{\infty}a_j x^{j/k} +\sum_{j=p}^{\infty}a_{j-p}\frac{\lceil(j+k-p)/k}{\lceil(j+k)/k}x^{j/k}=\frac{x^{(p+k)/k}}{\lceil(2+(p/k)} \tag{3.9a}$$

Putting $j=n-(k-p)$ in the 1st term, then

$$\sum_{j=1}^{\infty}a_{j+k-p}\frac{\lceil(j+(2k-p))/k}{\lceil(j+k)/k}x^{j/k}+\sum_{j=1}^{\infty}a_j x^{j/k} +\sum_{j=p}^{\infty}\frac{\lceil(j+k-p)/k}{\lceil(j+k)/k}a_{j-p}x^{j/k}=\frac{x^{(p+k)/k}}{\lceil 2+(p/k)} \tag{3.10}$$

From 1st and 2nd term, for $j=1,2,\cdots,p-1$,

$$a_{j+k-p}\frac{\lceil(j+(2k-p))/k}{\lceil(j+k)/k}+a_j=0 \tag{3.11}$$

From (3.9) and (3.10), we get for $j=1,2,\cdots,k-1$, $\ a_j=0$ i.e., $k=2$, $j=1$, $a_1=0$

From (3.10), we can write

$$\sum_{j=p}^{\infty}a_{j+k-p}\frac{\lceil(j+(2k-p))/k}{\lceil(j+k)/k}x^{j/k}+\sum_{j=p}^{\infty}a_j x^{j/k} +\sum_{j=p}^{\infty}a_{j-p}\frac{\lceil(j+k-p)/k}{\lceil(j+k)/k}x^{j/k}=\frac{x^{(p+k)/k}}{\lceil 2+(p/k)}$$

For $j \geq p$ (and $j \neq k+p$)

$$a_{j+k-p}\frac{\lceil (j+(2k-p))/k \rceil}{\lceil (j+k)/k \rceil} + a_j + a_{j-p}\frac{\lceil (j+k-p)/k \rceil}{\lceil (j+k)/k \rceil} = 0 \tag{3.12}$$

For $= k+p$,

$$a_{2k}\frac{\lceil 3 \rceil}{\lceil 2+(p/k) \rceil} + a_{k+p} + a_k\frac{\lceil 2 \rceil}{\lceil 2+(p/k) \rceil} = \frac{1}{\lceil 2+(p/k) \rceil} \tag{3.13}$$

From (3.12), $j = p$

$$a_k \cdot \frac{\lceil 2 \rceil}{\lceil (k+p)/k \rceil} + a_p + a_0\frac{\lceil 1 \rceil}{\lceil (k+p)/k \rceil} = 0 \tag{3.13a}$$

$\Rightarrow\ a_k = a_0$ as $a_p = 0$ from (3.11)

$$y(0) = a_0 = 1 \ \text{(given)} \ \Rightarrow\ a_k = -1 \tag{3.14}$$

For $j = p+i$ (from 3.12) $(j \neq k)$ and $i = 1, 2, 3, \cdots$

$$a_{k+i} \cdot \frac{\lceil (2k+i)/k \rceil}{\lceil (k+p+i)/k \rceil} + a_{p+i} + a_i\frac{\lceil (k+i)/k \rceil}{\lceil (k+p+i)/k \rceil} = 0 \tag{3.15}$$

From (3.15), $$a_{2+i} \cdot \frac{\lceil (4+i)/2 \rceil}{\lceil (3+i)/2 \rceil} + a_{1+i} + a_i\frac{\lceil (2+i)/2 \rceil}{\lceil (3+i)/2 \rceil} = 0$$

For $i = 1, 3, 4, 5, \cdots, (i \neq 2)$.

$$a_3 \cdot \frac{\lceil 5/2 \rceil}{\lceil 2 \rceil} + a_2 + 0 = 0 \ \Rightarrow\ a_3 = \frac{1}{\lceil 5/2 \rceil}$$

From (3.13), for $k = 2$, $$a_4\frac{\lceil 3 \rceil}{\lceil 5/2 \rceil} = a_3 + a_2\frac{\lceil 2 \rceil}{\lceil 5/2 \rceil} = \frac{1}{\lceil 5/2 \rceil} \ \Rightarrow\ a_4 = \frac{1}{2}$$

For $i = 3$ (from (3.15)): $$a_5\frac{\lceil 7/2 \rceil}{\lceil 3 \rceil} + a_4 + a_3\frac{\lceil 5/2 \rceil}{\lceil 3 \rceil} = 0 \ \Rightarrow\ a_5 = -\frac{2}{\lceil 7/2 \rceil}$$

For $i = 4$ (from (3.15)): $$a_6\frac{\lceil 4 \rceil}{\lceil 7/2 \rceil} + a_5 + a_4\frac{\lceil 3 \rceil}{\lceil 7/2 \rceil} = 0 \ \Rightarrow\ a_6 = \frac{1}{6}$$

For $i = 5$ (from (3.15)): $$a_7\frac{\lceil 9/2 \rceil}{\lceil 4 \rceil} + a_6 + a_5\frac{\lceil 7/2 \rceil}{\lceil 4 \rceil} = 0 \ \Rightarrow\ a_7 = \frac{1}{\lceil 9/2 \rceil}$$

For $i = 6$ (from (3.15)): $a_8 \frac{\lceil 5 \rceil}{\lceil 9/2 \rceil} + a_7 + a_6 \frac{\lceil 4 \rceil}{\lceil 9/2 \rceil} = 0 \Rightarrow a_8 = -\frac{2}{\lfloor 4 \rfloor} = -\frac{1}{12}$

For $i = 7$ (from (3.15)): $a_9 \frac{\lceil 11/2 \rceil}{\lceil 5 \rceil} + a_8 + a_7 \frac{\lceil 9/2 \rceil}{\lceil 5 \rceil} = 0 \Rightarrow a_9 = \frac{1}{\lceil 11/2 \rceil}$

For $i = 8$ (from (3.15)): $a_{10} \frac{\lceil 6 \rceil}{\lceil 11/2 \rceil} + a_9 + a_8 \frac{\lceil 5 \rceil}{\lceil 11/2 \rceil} = 0 \Rightarrow a_{10} = \frac{1}{\lfloor 5 \rfloor} = \frac{1}{\lceil 6 \rceil}$ and so on.

For $\alpha = \frac{1}{2}$

$$y_{\text{PSM}}(x) = 1 - x + \frac{x^{3/2}}{\lceil 5/2 \rceil} + \frac{x^2}{2!} - 2\frac{x^{5/2}}{\lceil 7/2 \rceil} + \frac{x^3}{3!} + \frac{x^{7/2}}{\lceil 9/2 \rceil} - 2\frac{x^4}{4!} + \frac{x^{9/2}}{\lceil 11/2 \rceil} + \cdots \tag{3.16}$$

(b) Adomian Decomposition Method. We consider the differential equation of the form

$$L(y(x)) + N(y(x)) = f(x) \tag{3.17}$$

where L is the (highest order) linear differential operator and N is a (in general) a nonlinear operator (which includes all other operators in differential equation other than L).

The function $f(x)$ is a known function of independent variable x and $y(x)$ is the unknown function which is the solution of (3.17).

ADM defines the unknown function $y(x)$ by an infinite series

$$y(x) = \sum_{i=0}^{\infty} y_i(x) \tag{3.18}$$

where the components $y_i(x)$ are determined using recursive relation. The non-linear operator $N(y(x))$ can be decomposed in an infinite series of Adomian polynomials given by

$$N(y) = \sum_{i=0}^{\infty} A_i(y_0(x), y_i(x), \cdots, y_1(x)) \tag{3.19}$$

where A_i's are called so-called Adomian polynomials of $y_0(x)$, $y_i(x), \cdots, y_1(x)$ and defined by

$$\begin{aligned} A_i &= \frac{1}{i!}\left(\frac{d^i}{d\lambda^i}\left[N\left(\sum_{j=0}^{i}\lambda^j y_j\right)\right]\right)_{\lambda=0} \quad \text{for } i = 0, 1, 2, \cdots \\ &= \frac{1}{i!}\left(\frac{d^i}{d\lambda^i}\left[N\left(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3 + \cdots\right)\right]\right)_{\lambda=0} \end{aligned} \tag{3.20}$$

Now, we consider the fractional ordinary differential equation:

$$\frac{dy}{dx} + \frac{{}^{c}d^{\alpha}y(x)}{dx^{\alpha}} + y(x) = x, \quad x \geq 0 \tag{3.21}$$

where $0 < \alpha < 1$ and $\dfrac{{}^{c}d^{\alpha}}{dx^{\alpha}} \equiv {}^{c}D^{\alpha}$ is the fractional differential operators of order 'α' as defined by the Caputo [12].

$y(0) = 1$ is the initial condition of equation (3.21).

Define $D \equiv \dfrac{d}{dx}$ and ${}^{c}D^{\alpha} \equiv \dfrac{d^{\alpha}}{dx^{\alpha}}$, then equation (3.16) can be written as

$$Dy(x) + {}^{c}D^{\alpha}y(x) + y(x) = x \tag{3.22}$$

In equation (3.22), we choose $L = D$ and

$$N(y(x)) \equiv {}^{c}D^{\alpha}y(x) + y(x) \quad \text{and} \quad f(x) = x. \tag{3.22a}$$

Then rewriting equation (3.22) in the form $L[y(x)] = -N(y(x)) + x$ where $L^{-1} \equiv I$.

Applying L^{-1} on both sides of equation (3.22), we get

$$y(x) - y(0) = \frac{x^2}{2} - L^{-1}[N(y(x))] \tag{3.23}$$

$$\text{or,} \quad y(x) = 1 + \frac{x^2}{2} - L^{-1}[N(y(x))] = y_0(x) - L^{-1}[N(y(x))].$$

Using (3.18) and (3.19), equation (3.23) can be written as

$$y(x) = \sum_{i=0}^{\infty} y_i(x) = y_0(x) - L^{-1}\left(\sum_{i=0}^{\infty} A_i(y_0(x), y_1(x), \cdots, y_i(x))\right) \tag{3.24}$$

From equation (3.24),

$$y_i(x) = -L^{-1}\left(A_{i-1}(y_0(x), y_1(x), \cdots, y_{i-1}(x))\right), \tag{3.25}$$

where

$$A_{i-1} = \frac{1}{(i-1)!}\left(\frac{d^{i-1}}{d\lambda^{i-1}}\left[N\left(\sum_{j=0}^{i-1}\lambda^{j}y_j\right)\right]\right)_{\lambda=0} \quad \text{for } i = 1, 2, 3, \cdots$$

For $i = 1$,

$$\begin{aligned} y_1(x) &= -L^{-1}(A_0(y_0)) = L^{-1}\left[{}^cD^{\alpha}y_0(x) + y_0(x)\right] \\ &= -I^{1-\alpha}\left[y_0(x) - y_0(0)\right] - Iy_0(x) \\ &= -\frac{x^{3-\alpha}}{\lceil 4-\alpha} - \left(x + \frac{x^3}{6}\right) \end{aligned}$$

For $\alpha = \frac{1}{2}$,

$$y_1(x) = -x - \frac{x^{5/2}}{\lceil 7/2} - \frac{x^3}{3!}$$

From (3.25),

$$\begin{aligned} y_2(x) &= -(I^{1-\alpha} + I)y_1(x) \\ &= \frac{x^{4-2\alpha}}{\lceil 5-2\alpha} + \frac{x^{2-\alpha}}{\lceil 3-\alpha} + 2\frac{x^{4-\alpha}}{\lceil 5-\alpha} + \frac{x^4}{4!} + \frac{x^2}{2!} \end{aligned} \tag{3.25a}$$

For $\alpha = \frac{1}{2}$,

$$y_2(x) = \frac{x^{3/2}}{\lceil 5/2} + \frac{x^2}{2!} + \frac{x^3}{3!} + 2\frac{x^{7/2}}{\lceil 9/2} + \frac{x^4}{4!}$$

$$\begin{aligned} y_3(x) &= -(I^{1-\alpha} + I)y_2(x), \quad y_2(0) = 0 \\ &= -\left[\frac{x^{5-3\alpha}}{\lceil 6-3\alpha} + 3\frac{x^{5-2\alpha}}{\lceil 6-2\alpha} + 2\frac{x^{3-\alpha}}{\lceil 4-\alpha} + 2\frac{x^{5-\alpha}}{\lceil 6-\alpha} + \frac{x^{3-2\alpha}}{\lceil 4-2\alpha} + \frac{x^3}{3!} + \frac{x^5}{5!}\right] \end{aligned}$$

For $\alpha = \frac{1}{2}$,

$$\begin{aligned} y_3(x) &= -\left[\frac{x^2}{2!} + 2\frac{x^{5/2}}{\lceil 7/2} + \frac{x^3}{3!} + \frac{x^{7/2}}{\lceil 9/2} + 3\frac{x^4}{4!} + 3\frac{x^{9/2}}{\lceil 11/2} + \frac{x^5}{5!}\right] \\ y_4(x) &= -(I^{1-\alpha} + I)y_3(x), \quad y_3(0) = 0 \\ &= \frac{x^{6-4\alpha}}{\lceil 7-4\alpha} + 4\frac{x^{6-3\alpha}}{\lceil 7-3\alpha} + 6\frac{x^{6-2\alpha}}{\lceil 7-2\alpha} + 4\frac{x^{6-\alpha}}{\lceil 7-\alpha} \\ &\quad + 3\frac{x^{4-2\alpha}}{\lceil 5-2\alpha} + 3\frac{x^{4-\alpha}}{\lceil 5-\alpha} + \frac{x^{4-3\alpha}}{\lceil 5-3\cdot\alpha} + \frac{x^4}{4!} + \frac{x^6}{6!} \end{aligned}$$

For $\alpha = \frac{1}{2}$,

$$y_4(x) = \frac{x^{5/2}}{\lceil 7/2 \rceil} + \frac{3x^3}{3!} + \frac{3x^{7/2}}{\lceil 9/2 \rceil} + \frac{2x^4}{4!} + \frac{4x^{9/2}}{\lceil 11/2 \rceil} + \frac{4x^{11/2}}{\lceil 13/2 \rceil} + \frac{6x^5}{5!} + \frac{x^6}{6!}$$

$$y_5(x) = -(I^{1-\alpha} + I)y_4(x)$$

$$= -\left[\frac{x^{7-5\alpha}}{\lceil 8-5\alpha \rceil} + 5\frac{x^{7-4\alpha}}{\lceil 8-4\alpha \rceil} + 10\frac{x^{7-3\alpha}}{\lceil 8-3\alpha \rceil} + 10\frac{x^{7-2\alpha}}{\lceil 8-2\alpha \rceil} + 5\frac{x^{7-\alpha}}{\lceil 8-\alpha \rceil}\right.$$

$$\left. + 4\frac{x^{5-3\alpha}}{\lceil 6-3\alpha \rceil} + 6\frac{x^{5-2\alpha}}{\lceil 6-2\alpha \rceil} + 4\frac{x^{5-\alpha}}{\lceil 6-\alpha \rceil} + \frac{x^{5-4\alpha}}{\lceil 6-4\alpha \rceil} + \frac{x^5}{5!} + \frac{x^7}{7!}\right]$$

For $\alpha = \frac{1}{2}$,

$$y_5(x) = -\left[\frac{4x^{7/2}}{\lceil 9/2 \rceil} + \frac{x^3}{3!} + \frac{6x^4}{4!} + \frac{5x^{9/2}}{\lceil 11/2 \rceil} + \frac{6x^5}{5!} + \frac{10x^{11/2}}{\lceil 13/2 \rceil} + \frac{10x^6}{6!} + 5\frac{x^{13/2}}{\lceil 15/2 \rceil} + \frac{x^7}{7!}\right]$$

$$y_6(x) = -(I^{1-\alpha} + I)y_5(x)$$

$$y_6(x) = \frac{x^{8-6\alpha}}{\lceil 9-6\alpha \rceil} + 6\frac{x^{8-5\alpha}}{\lceil 9-5\alpha \rceil} + 15\frac{x^{8-4\alpha}}{\lceil 9-4\alpha \rceil} + 20\frac{x^{8-2\alpha}}{\lceil 9-3\alpha \rceil} + 15\frac{x^{8-2\alpha}}{\lceil 9-2\alpha \rceil} + \frac{x^{6-5\alpha}}{\lceil 7-5\alpha \rceil}$$

$$+5\frac{x^{6-4\alpha}}{\lceil 7-4\alpha \rceil} + 10\frac{x^{6-3\alpha}}{\lceil 7-3\alpha \rceil} + 10\frac{x^{6-2\alpha}}{\lceil 7-2\alpha \rceil} + 5\frac{x^{6-\alpha}}{\lceil 7-\alpha \rceil} + 5\frac{x^{8-\alpha}}{\lceil 9-\alpha \rceil} + \frac{x^6}{6!} + \frac{x^8}{8!}$$

For $\alpha = \frac{1}{2}$,

$$y_6(x) = \frac{x^{7/2}}{\lceil 9/2 \rceil} + \frac{5x^4}{4!} + \frac{10x^{9/2}}{\lceil 11/2 \rceil} + \frac{11x^5}{5!} + \frac{11x^{11/2}}{\lceil 13/2 \rceil} + \frac{16x^6}{6!} + \frac{20x^{13/2}}{\lceil 15/2 \rceil}$$

$$+\frac{15x^7}{7!} + 5\frac{x^{15/2}}{\lceil 17/2 \rceil} + \frac{x^8}{8!}$$

$$y_7(x) = -(I^{1-\alpha} + I)y_6(x)$$

$$= -\left[\frac{x^{9-7\alpha}}{\lceil 10-7\alpha \rceil} + 7\frac{x^{9-6\alpha}}{\lceil 10-6\alpha \rceil} + 21\frac{x^{9-5\alpha}}{\lceil 10-5\alpha \rceil} + 35\frac{x^{9-4\alpha}}{\lceil 10-4\alpha \rceil} + 35\frac{x^{9-3\alpha}}{\lceil 10-3\alpha \rceil}\right.$$

$$+20\frac{x^{9-2\alpha}}{\lceil 10-2\alpha \rceil} + \frac{x^{7-6\alpha}}{\lceil 8-6\alpha \rceil} + 6\frac{x^{7-5\alpha}}{\lceil 8-5\alpha \rceil} + 15\frac{x^{7-4\alpha}}{\lceil 8-4\alpha \rceil} + 20\frac{x^{7-3\alpha}}{\lceil 8-3\alpha \rceil}$$

$$\left. +15\frac{x^{7-2\alpha}}{\lceil 8-2\alpha \rceil} + 6\frac{x^{7-\alpha}}{\lceil 8-\alpha \rceil} + 6\frac{x^{9-\alpha}}{\lceil 10-\alpha \rceil} + \frac{x^7}{7!} + \frac{x^9}{9!}\right]$$

For $\alpha = \frac{1}{2}$,

$$y_7(x) = -\frac{x^4}{4!} - 6\frac{x^{9/2}}{\overline{|11/2}} - \frac{15x^5}{5!} - 21\frac{x^{11/2}}{\overline{|13/2}}$$
$$y_8(x) = \frac{x^{10-8\alpha}}{\overline{|11-8\alpha}} + \frac{x^{10-7\alpha}}{\overline{|11-7\alpha}} + 7\frac{x^{10-7\alpha}}{\overline{|11-7\alpha}} + 7\frac{x^{10-6\alpha}}{\overline{|11-6\alpha}}$$

For $\alpha = \frac{1}{2}$,

$$y_8(x) = \frac{x^{9/2}}{\overline{|11/2}} + \cdots$$
$$y_{ADM}(x) = 1 - x + \frac{x^{3/2}}{\overline{|5/2}} + \frac{x^2}{2!} - 2\frac{x^{5/2}}{\overline{|7/2}} + \frac{x^3}{3!} + \frac{x^{7/2}}{\overline{|9/2}} - 2\frac{x^4}{4!} + \frac{x^{9/2}}{\overline{|11/2}} + \cdots \quad (3.26)$$

c) Homotopy Analysis Method (HAM). Let us consider $L \equiv \dfrac{d}{dx} \equiv D$ and $I \equiv D^{-1}$.

The differential equation can be written as $N(y(x)) = Dy(x) + {}^cD^\alpha y(x) + y(x) - x = 0$

Then

$$N(\phi(x,q)) = D\phi(x,q) + {}^cD^\alpha\phi(x,q) + \phi(x,q) - x = 0 \quad (3.27)$$

Deformation equations :

$$L\left[y_m(x) - \lambda_m y_{m-1}(x)\right] = hR_m\left[\overline{y_{m-1}}(x)\right] \quad (3.28)$$

where

$$R_m\left[y_{m-1}, y_{m-2}, \cdots, y_2, y_1, y_0\right] = \frac{1}{(m-1)!}\left\{\frac{\partial^{m-1}}{\partial q^{m-1}}N[\phi(x,q)]\right\}_{q=0}$$

Here $\lambda_m = \begin{cases} 0 & \text{for } m = 1 \\ 1 & \text{for } m \geq 2 \end{cases}$,

Then

$$\frac{\partial^{m-1}}{\partial q^{m-1}}N[\phi(x,q)] = D\left(\frac{\partial^{m-1}}{\partial q^{m-1}}\phi(x,q)\right) + {}^cD^\alpha\left(\frac{\partial^{m-1}}{\partial q^{m-1}}\phi(x,q)\right) + \frac{\partial^{m-1}}{\partial q^{m-1}}\phi(x,q) - \frac{\partial^{m-1}}{\partial q^{m-1}}(x)$$

Expanding $\phi(x,q)$ in terms of q:

$$\phi(x,q) = \phi(x,0) + \sum_{m=1}^{\infty} \left(\frac{\partial^m \phi}{\partial q^m}\right)_{q=0} \cdot \frac{q^m}{m!}$$

$$= y_0(x) + \sum_{m=1}^{\infty} y_m(x) q^m = \sum_{m=0}^{\infty} y_m(x) q^m \tag{3.29}$$

where

$$y_m(x) = \frac{1}{m!} \left(\frac{\partial^m \phi}{\partial q^m}\right)_{q=0} \quad \text{for } m = 0, 1, 2, \cdots$$

For $q = 1$, $\phi(x,1) = y(x) = \sum_{y=0}^{\infty} y_m(x)$

For HAM,

$$\phi(x,0) = y_0(x) \quad \text{and} \quad \phi(x,1) = y(x) \tag{3.30}$$

From (3.29),

$$\left(\frac{\partial^{m-1}}{\partial q^{m-1}}\right)_{q=0} = (m-1)! y_{m-1}(x)$$

$$\therefore \quad y_{m-1}(x) = \frac{1}{(m-1)!} \left(\frac{\partial^{m-1} \phi}{\partial q^{m-1}}\right)_{q=0}$$

Then

$$R_m = D\left(\frac{1}{(m-1)!}\left(\frac{\partial^{m-1}[\phi(x,q)]}{\partial q^{m-1}}\right)\right)_{q=0} + {}^{c}D^{\alpha}\left(\frac{1}{(m-1)!}\left(\frac{\partial^{m-1}[\phi(x,q)]}{\partial q^{m-1}}\right)\right)_{q=0}$$

$$+ \left(\frac{1}{(m-1)!}\left(\frac{\partial^{m-1}[\phi(x,q)]}{\partial q^{m-1}}\right)\right)_{q=0} - (1-\lambda_m)x$$

$$= Dy_{m-1} + {}^{c}D^{\alpha} y_{m-1} + y_{m-1} - (1-\lambda_m)x \quad \text{for } m = 1, 2, 3, \cdots \tag{3.31}$$

From (3.28):

$$L[y_m(x) - \lambda_m y_{m-1}(x)] = h\left[Dy_{m-1} + {}^{c}D^{\alpha} y_{m-1} + y_{m-1} - (1-\lambda_m)x\right] \tag{3.32}$$

$$\text{for } m = 1, 2, 3, \cdots$$

For $m = 1$,

$$L[y_1(x) - \lambda_1 y_0(x)] = h\left[Dy_0(x) + {}^{c}D^{\alpha} y_0(x) + y_0(x) - x\right] \tag{3.32a}$$

Let $y_0(x) = 1$

$$L[y_1(x)] = h[y_0(x) - x] = h(1 - x)$$
$$\therefore \quad y_1(x) = hI[y_0(x) - x] = h(x - x^2/2)$$

For $m = 2$, we get from (3.32),

$$L[y_2(x) - \lambda_2 y_1(x)] = (1 + h)y_1(x) + h^2(I^{1-\alpha} + I)(x - x^2/2)$$
$$= (1 + h)y_1(x) + h^2 \left[\frac{x^{2-\alpha}}{\lceil 3 - \alpha} + \frac{x^2}{2} - \frac{x^{3-\alpha}}{\lceil 4 - \alpha} - \frac{x^3}{3!} \right]$$

For $h = -1$, $\alpha = \frac{1}{2}$,

$$y_2(x) = \frac{x^{3/2}}{\lceil 5/2} + \frac{x^2}{2!} - \frac{x^{5/2}}{\lceil 7/2} - \frac{x^3}{3!}$$
$$y_3(x) = (1 + h)y_2(x) + h(I^{1-\alpha} + I)y_2(x)$$
$$= (1 + h)[\cdot\cdot\cdot\cdot\cdot] + h^3 \left[\frac{x^{3-2\alpha}}{\lceil 4 - 2\alpha} + \frac{x^{3-\alpha}}{\lceil 4 - \alpha} + \frac{x^{3-\alpha}}{\lceil 4 - \alpha} + \frac{x^3}{3!} \right.$$
$$\left. - \frac{x^{4-2\alpha}}{\lceil 5 - 2\alpha} - \frac{x^{4-\alpha}}{\lceil 5 - \alpha} - \frac{x^{4-\alpha}}{\lceil 5 - \alpha} - \frac{x^4}{4!} \right]$$

For $h = -1$, $\alpha = \frac{1}{2}$,

$$y_3(x) = -\frac{x^2}{2!} - 2\frac{x^{5/2}}{\lceil 7/2} + 2\frac{x^{7/2}}{\lceil 9/2} + \frac{x^4}{4!}$$
$$y_4(x) = (1 + h)[\cdot\cdot\cdot\cdot\cdot] + h^4 \left(\frac{x^{4-3\alpha}}{\lceil 5 - 3\alpha} + \frac{x^{4-2\alpha}}{\lceil 5 - 2\alpha} + 2\frac{x^{4-2\alpha}}{\lceil 5 - 2\alpha} + 2\frac{x^{4-\alpha}}{\lceil 5 - \alpha} + \frac{x^{4-\alpha}}{\lceil 5 - \alpha} \right.$$
$$\left. + \frac{x^4}{4!} - \frac{x^{5-3\alpha}}{\lceil 6 - 3\alpha} - \frac{x^{5-2\alpha}}{\lceil 6 - 2\alpha} - 2\frac{x^{5-2\alpha}}{\lceil 6 - 2\alpha} - 2\frac{x^{5-\alpha}}{\lceil 6 - \alpha} - \frac{x^{5-\alpha}}{\lceil 6 - \alpha} - \frac{x^5}{5!} \right)$$

For $h = -1$, $\alpha = \frac{1}{2}$,

$$y_4(x) = \frac{x^{5/2}}{\lceil 7/2} + \frac{3x^3}{3!} + 2\frac{x^{7/2}}{\lceil 9/2} - \frac{2x^4}{4!} - \frac{3x^{9/2}}{\lceil 11/2} - \frac{x^5}{5!}$$
$$y_5(x) = (1 + h)[\cdot\cdot\cdot\cdot\cdot] + h^5 \left[\frac{x^{5-4\alpha}}{\lceil 6 - 4\alpha} + 4\frac{x^{5-3\alpha}}{\lceil 6 - 3\alpha} + 6\frac{x^{5-2\alpha}}{\lceil 6 - 2\alpha} + 4\frac{x^{5-\alpha}}{\lceil 6 - \alpha} + \frac{x^5}{5!} \right.$$
$$\left. - \frac{x^{6-4\alpha}}{\lceil 7 - 4\alpha} - 4\frac{x^{6-3\alpha}}{\lceil 7 - 3\alpha} - 6\frac{x^{6-2\alpha}}{\lceil 7 - 2\alpha} - 4\frac{x^{6-\alpha}}{\lceil 7 - \alpha} - \frac{x^6}{6!} \right]$$

For $h=-1$, $\alpha=\frac{1}{2}$,

$$y_5(x) = -\frac{x^3}{3!} - 4\frac{x^{7/2}}{\lceil 9/2\rceil} - 5\frac{x^4}{4!} + 5\frac{x^5}{5!} + 4\frac{x^{11/2}}{\lceil 13/2\rceil} + \frac{x^6}{6!}$$

$$\begin{aligned} y_6(x) = (1+h)[\cdots\cdots] + h^6\Bigg[& \frac{x^{6-5\alpha}}{\lceil 7-5\alpha\rceil} + 5\frac{x^{6-4\alpha}}{\lceil 7-4\alpha\rceil} + 10\frac{x^{6-3\alpha}}{\lceil 7-3\alpha\rceil} + 10\frac{x^{6-2\alpha}}{\lceil 7-2\alpha\rceil} \\ & + 5\frac{x^{6-\alpha}}{\lceil 7-\alpha\rceil} + \frac{x^6}{6!} - \frac{x^{7-5\alpha}}{\lceil 8-5\alpha\rceil} - 5\frac{x^{7-4\alpha}}{\lceil 8-4\alpha\rceil} - 10\frac{x^{7-3\alpha}}{\lceil 8-3\alpha\rceil} \\ & - 10\frac{x^{7-2\alpha}}{\lceil 8-2\alpha\rceil} - 5\frac{x^{7-\alpha}}{\lceil 8-\alpha\rceil} - \frac{x^7}{7!}\Bigg] \end{aligned}$$

For $h=-1$, $\alpha=\frac{1}{2}$,

$$y_6(x) = \frac{x^{7/2}}{\lceil 9/2\rceil} + 5\frac{x^4}{4!} + 9\frac{x^{9/2}}{\lceil 11/2\rceil} + 5\frac{x^5}{5!} - 5\frac{x^{11/2}}{\lceil 13/2\rceil} - 9\frac{x^6}{6!} - 5\frac{x^{13/2}}{\lceil 15/2\rceil} - \frac{x^7}{7!}$$

$$\begin{aligned} y_7(x) = (1+h)[\cdots\cdots] + h^7\Bigg[& \frac{x^{7-6\alpha}}{\lceil 8-6\alpha\rceil} + 6\frac{x^{7-5\alpha}}{\lceil 8-5\alpha\rceil} + 15\frac{x^{7-4\alpha}}{\lceil 8-4\alpha\rceil} \\ & + 10\frac{x^{7-3\alpha}}{\lceil 8-3\alpha\rceil} + 15\frac{x^{7-2\alpha}}{\lceil 8-2\alpha\rceil} + \cdots\Bigg] \end{aligned}$$

For $h=-1$, $\alpha=\frac{1}{2}$

$$y_7(x) = -\frac{x^4}{4!} - 6\frac{x^{9/2}}{\lceil 11/2\rceil} - 14\frac{x^5}{5!} + \cdots$$

$$y_8(x) = (1+h)[\cdots\cdots] + h^8\left[\frac{x^{8-7\alpha}}{\lceil 9-7\alpha\rceil} + \frac{x^{8-6\alpha}}{\lceil 9-6\alpha\rceil} + 6\frac{x^{8-6\alpha}}{\lceil 9-6\alpha\rceil} + 6\frac{x^{8-5\alpha}}{\lceil 9-5\alpha\rceil} + \cdots\right]$$

For $h=-1$, $\alpha=\frac{1}{2}$,

$$y_8(x) = \frac{x^{9/2}}{\lceil 11/2\rceil} + 7\frac{x^6}{6!} + \cdots$$

$$y_{\mathrm{HAM}}(x) = 1 - x + \frac{x^{3/2}}{\lceil 5/2\rceil} + \frac{x^2}{2!} - 2\frac{x^{5/2}}{\lceil 7/2\rceil} + \frac{x^3}{3!} + \frac{x^{7/2}}{\lceil 9/2\rceil} - 2\frac{x^4}{4!} + \frac{x^{9/2}}{\lceil 11/2\rceil} + \cdots \quad (3.33)$$

4. Comparison of three methods used in solving a nonlinear fractional differential Equation :

(A) Power Series Method (PSM) : Let us consider a non-linear fractional differential equation,

$$\frac{dy}{dx} - {}^cD^{1/2}y - y^2 = 2x, \quad x \geq 0 \quad \text{and} \quad y(0) = 1 \tag{4.1}$$

Let

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n/2} \tag{4.2}$$

be the power series solution of the fractional ordinary **non-linear** differential equation (4.1).

Applying $I^{1/2}$ on both sides of (4.1),

$$I^{1/2}Dy - (I^{1/2}\,{}^cD^{1/2})y - I^{1/2}y^2 = \frac{8}{3\sqrt{17}}x^{3/2} \tag{4.3}$$

Here ${}^cD^{1/2}$ is the **Caputo** derivative of order $1/2$ and $I^{1/2}$ is the R-L integral of order $1/2$.

We know that $I^{1/2}\,{}^cD^{1/2}f(x) = f(x) - \sum_{k=0}^{m-1} f^k(0+)\frac{x^k}{k!}$, $x > 0$ and $m-1 \leq \alpha < m$ is the order of fractional derivative.

Here $\alpha = 1/2$ and $m = 1$,

Now, if $f(x) = 1$, then $I^{1/2}\,{}^cD^{1/2}f(x) = 1 - 1 = 0$

Again, if $f(x)$ is a function of x, such that $f(0) = 0$, then $I^{1/2}\,{}^cD^{1/2}f(x) = f(x) - f(0) = f(x)$.

Substituting (4.2) in (4.3), we get,

$$\sum_{n=1}^{\infty} \frac{n}{2} \frac{\lceil n/2}{\lceil n/2 + 1/2} a_n x^{\frac{n}{2}-\frac{1}{2}} - \left\{ \sum_{n=0}^{\infty} a_n x^{n/2} - a_0 \right\}$$

$$-I^{1/2} \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} a_{n-1-i} a_i x^{\frac{n-1-i}{2}} x^{\frac{i}{2}} = \frac{2}{\lceil 5/2} x^{3/2} \tag{4.4}$$

In the last term, put $n = j+1$ ($n = 1, j = 0$) and in the other two term replacing n by j (after integration)

$$\sum_{j=0}^{\infty} \left(\frac{j+1}{2}\right) \frac{\lceil (j+1)/2}{\lceil (j+2)/2} a_{j+1} x^{j/2} - \sum_{j=1}^{\infty} a_j x^{j/2}$$

$$-\sum_{j=1}^{\infty}\sum_{i=0}^{j-1}\frac{\lceil(j+1)/2\rceil}{\lceil(j+2)/2\rceil}a_{j-1-i}a_i x^{\frac{i}{2}} = 2\frac{x^{3/2}}{5/2} \tag{4.5}$$

$$y(0) = a_0 = 1 \;\Rightarrow\; a_0 = 1.$$

For $j=0$, $\frac{1}{2}\frac{\lceil 1/2\rceil}{\lceil 1\rceil}a_1 = 0 \;\Rightarrow\; a_1 = 0$

For $j=1$, $\frac{1\cdot\lceil 1\rceil}{\lceil 3/2\rceil}a_2 - a_1 - a_0^2\frac{\lceil 1\rceil}{\lceil 3/2\rceil} = 0 \;\Rightarrow\; a_2 = a_0^2 = 1$

For $j=2$, $\frac{(3/2)\lceil 3/2\rceil}{\lceil 2\rceil}a_3 - a_2 - \sum_{i=0}^{1}a_{1-i}a_i\frac{\lceil 3/2\rceil}{\lceil 2\rceil} = 0 \;\Rightarrow\; a_3 = \frac{1}{\lceil 5/2\rceil}.$

For $j=3$, from (4.5), $\frac{2\lceil 2\rceil}{\lceil 5/2\rceil}a_4 - a_3 - \sum_{i=0}^{2}\frac{\lceil 2\rceil}{\lceil 5/2\rceil}a_{3-1-i}a_i = \frac{\lceil 2\rceil}{\lceil 5/2\rceil} \;\Rightarrow\; a_4 = \frac{5}{2}$

In general for $j \geq 4$,

$$\left(\frac{j+1}{2}\right)\frac{\lceil(j+1)/2\rceil}{\lceil(j+2)/2\rceil}a_{j+1} - a_j - \sum_{i=0}^{j-1}a_{j-1-i}a_i\frac{\lceil(j+1)/2\rceil}{\lceil(j+2)/2\rceil} = 0$$

$$\Rightarrow\; a_{j+1} = \frac{\lceil(j+2)/2\rceil}{\lceil(j+3)/2\rceil}a_j + \frac{2}{j+1}\sum_{i=0}^{j-1}a_{j-1-i}a_i$$

For $j=4$, $a_5 = \frac{\lceil 3\rceil}{\lceil 7/2\rceil}a_4 + \frac{2}{5}\sum_{i=0}^{3}(a_{3-i}a_i) \;\Rightarrow\; a_5 = \frac{56}{15\sqrt{\pi}}$

For $j=5$, $a_6 = \frac{\lceil 7/2\rceil}{\lceil 4\rceil}a_5 + \frac{1}{3}\sum_{i=1}^{4}(a_{4-i}a_i) = \frac{19}{6}$

For $j=6$, $a_7 = \frac{\lceil 4\rceil}{\lceil 9/2\rceil}a_6 + \frac{2}{7}\sum_{i=0}^{5}(a_{5-i}a_i) = \frac{608}{105\sqrt{\pi}}$

For $j=7$, $a_8 = \frac{\lceil 9/2\rceil}{\lceil 5\rceil}a_7 + \frac{1}{4}\sum_{i=0}^{6}(a_{6-i}a_i) = \frac{53}{12} + \frac{9}{4\pi}$

For $j=8$, $a_9 = \frac{\lceil 5\rceil}{\lceil 11/2\rceil}a_8 + \frac{2}{9}\sum_{i=0}^{7}(a_{7-i}a_i) = \frac{256}{135\sqrt{\pi}} + \frac{1024}{2835\pi^{3/2}}$

$\therefore$ General power series solution is:

$$y_{PSM} = 1 + x + \frac{4}{3\sqrt{\pi}}x^{3/2} + \frac{5}{2}x^2 + \frac{56}{15\sqrt{\pi}}x^{5/2} + \frac{19}{6}x^3 + \frac{608}{105\sqrt{\pi}}x^{7/2} + \left(\frac{53}{12} + \frac{4}{9\pi}\right)x^4 + \left(\frac{1256}{135\sqrt{\pi}} + \frac{1024}{2835\pi^{3/2}}\right)x^{9/2} + \cdots$$

(B) Adomian Decomposition Method (ADM). We consider the non-linear fractional differential equation:

$$L[y(x)] + N[y(x)] = f(x)$$

where L is a linear operator and N is a non-linear operator and $f(x)$ is a known function of x.

ADM defines the unknown function $y(x)$ by an infinite series:

$$y(x) = \sum_{i=0}^{\infty} y_i(x) \tag{4.6}$$

where the components $y_i(x)$ are determined by using recurrence relations.

The non-linear part $N[y(x)]$ can be decomposed in an infinite series of Adomian Polynomials given by

$$N(y) = \sum_{i=0}^{\infty} A_i(y_0, y_1, \cdots, y_i) \tag{4.7}$$

where A_i's are the so called Adomian Polynomials and defined by

$$A_i(y_0, y_1, \cdots, y_i) = \frac{1}{i!}\left\{\frac{d^i}{d\lambda^i}\left[N\left(\sum_{j=0}^{i}\lambda^j y_j(x)\right)\right]\right\}_{\lambda=0} \qquad i = 1, 2, 3, \cdots \tag{4.8}$$

Explicitly (4.8) can be written as, for $N(y(x)) = {}^cD^{1/2}y(x) + (y(x))^2$

$$\left.\begin{aligned} A_0 &= N(y_0) = y_0^2 + {}^cD^{1/2}y_0(x) \\ A_1 &= y_1 N'(y_0) = 2y_0y_1 + {}^cD^{1/2}y_1(x) \\ A_2 &= y_2N'(y_0) + \tfrac{1}{2}y_1^2N''(y_0) = 2y_0y_2 + y_1^2 + {}^cD^{1/2}y_2 \\ A_3 &= y_3N'(y_0) + y_1y_2N''(y_0) + \tfrac{1}{3}y_1^3N'''(y_0) = 2y_0y_3 + 2y_1y_2 + {}^cD^{1/2}y_3 \end{aligned}\right\} \tag{4.8a}$$

We will apply ADM for our problems:

$$Dy(x) - {}^cD^{1/2}y(x) - y^2(x) = 2x, \quad x \geq 0 \tag{4.9}$$

with initial condition

$$y(0) = 1 \tag{4.10}$$

Rewriting equation (4.9),

$$L[y(x)] = N[y(x)] + 2x \tag{4.11}$$

where we consider $L \equiv \frac{d}{dx}$ and $N(y(x)) \equiv {}^cD^{1/2}y(x) + (y(x))^2$ in equation (4.11)

Applying L^{-1} on both sides of (4.11), we get

$$y(x) = (y(0) + x^2) + L^{-1}\left[N(y(x))\right] \tag{4.12}$$

Let us choose,

$$y_0(x) = y(0) + x^2 = 1 + x^2 \tag{4.12a}$$

Then after using (4.6) and (4.7),we get

$$y_{i+1}(x) = L^{-1}\left[A_i(y_0, y_1, \cdots, y_i)\right], \quad \text{for } i = 0, 1, 2, \cdots \tag{4.13}$$

where A_i's are given by (4.8) or (4.8a).

Here $L^{-1} = I$, integral operator and $A_0 = {}^cD^{1/2}y_0(x) + y_0^2(x)$

$$\begin{aligned} \therefore \quad y_1(x) &= I\left[{}^cD^{1/2}y_0(x)\right] + I\left[y_0^2(x)\right] \\ &= x + \frac{2}{\sqrt{7/2}}\, x^{5/2} + \frac{2}{3}x^3 + \frac{x^5}{5} \end{aligned}$$

For $= 1$ in (4.13):

$$\begin{aligned} y_2(x) &= L^{-1}\left[A_1(y_0, y_1)\right] = I^{1/2}\left[y_1(x) - y_1(0) + 2I(y_0 y_1)\right] \\ &= \frac{x^{3/2}}{\sqrt{5/2}} + x^2 + \frac{x^3}{3} + \frac{16}{2\sqrt{7/2}}x^{7/2} + \frac{5}{6}x^4 + \frac{232}{231\sqrt{7/2}}x^{11/2} + \frac{13}{45}x^6 + \frac{x^8}{20} \end{aligned}$$

For $i = 2$, from (4.13)

$$\begin{aligned} y_3(x) &= I^{1/2}(y_2(x)) + 2I(y_0 y_2) + I(y_1^2) \\ &= \frac{x^2}{2} + \frac{4}{\sqrt{7/2}}x^{5/2} + x^3 + \frac{2}{\sqrt{9/2}}x^{7/2} + \frac{x^4}{2} + \frac{135}{2\sqrt{11/2}}x^{9/2} \end{aligned}$$

$$+x^5+\left(\frac{167}{720}+\frac{2}{3(\lceil 7/2\rceil)^2}\right)x^6+\frac{1009}{\lceil 15/2\rceil}x^{13/2}+\frac{139}{315}x^7$$
$$+\left(\frac{98680}{17017\times 15\lceil 7/2\rceil}\right)x^{17/2}+\frac{17}{162}x^9+\frac{7}{550}x^{11}$$

For $i=3$, from (4.13)

$$\begin{aligned}y_4(x) &= I^{1/2}(y_3)+2I(y_0y_3)+2I(y_1y_2)\\ &= \frac{x^{5/2}}{\lceil 7/2\rceil}+x^3+\left(\frac{26}{7\lceil 7/2\rceil}+\frac{6}{\lceil 9/2\rceil}\right)x^{7/2}+\frac{13}{12}x^4+\frac{16}{\lceil 11/2\rceil}x^{9/2}\\ &\quad+\frac{2516}{231\lceil 7/2\rceil}x^{11/2}+\left(\frac{263}{240}+\frac{4}{5\lceil 5/2\rceil\lceil 7/2\rceil}\right)x^5+\frac{7}{6}x^6+\cdots\end{aligned}$$

For $i=4$, from (4.13)

$$\begin{aligned}y_5(x) &= I^{1/2}(y_4)+I(2y_0y_4+2y_1y_3+y_2^2)\\ &= \frac{x^3}{3!}+\frac{8}{\lceil 9/2\rceil}x^{7/2}+\left(\frac{13}{8}+\frac{1}{9\pi}\right)x^4+\frac{247\times 16}{9\times 105\sqrt{\pi}}x^{9/2}+\cdots\end{aligned}$$

Similarly,

$$\begin{aligned}y_6(x) &= \frac{x^{7/2}}{\lceil 9/2\rceil}+\left(\frac{8}{\lceil 11/2\rceil}+\frac{2}{5}\right)x^4+\left[\left(\frac{13}{8}+\frac{1}{9\pi}\right)\frac{\lceil 5\rceil}{\lceil 11/2\rceil}+\frac{16}{\lceil 11/2\rceil}\right.\\ &\quad\left.+\frac{1}{\lceil 11/2\rceil}+\frac{1}{9\lceil 5/2\rceil}\right]x^{9/2}+\cdots\\ \therefore\ y_{ADM}(x) &= 1+x+\frac{4}{3\sqrt{\pi}}x^{3/2}+\frac{5}{2}x^2+\frac{56}{15\sqrt{\pi}}x^{5/2}+\frac{19}{6}x^3+\frac{608}{105\sqrt{\pi}}x^{7/2}\\ &\quad+\left(\frac{53}{12}+\frac{9}{4\pi}\right)x^4+\left(\frac{1256}{135\sqrt{\pi}}+\frac{1024}{2835\pi^{3/2}}\right)x^{9/2}+\cdots\end{aligned}$$

(C) Homotopy Analysis Method (HAM). We consider the same differential equation

$$Dy(x)-{}^cD^{1/2}y(x)-(y(x))^2=2x \tag{4.14}$$

with initial condition $y(0)=1$.

$$\text{or, }\ N[y(x)]=D[y(x)]-{}^cD^{1/2}[y(x)]-y^2(x)-2x=0 \tag{4.15}$$

Using HAM we get,

$$L[y_m(x) - \lambda_m y_{m-1}(x)] = hR_m[\overline{y_{m-1}}(x)] \tag{4.16}$$

where $\lambda_m = \begin{cases} 0 & \text{for } m = 1 \\ 1 & \text{for } m \geq 2 \end{cases}$ and

$$R_m\left[y_{m-1}, y_{m-2}, \cdots, y_2, y_1, y_0\right] = \frac{1}{(m-1)!}\left\{\frac{\partial^{m-1}}{\partial x^{m-1}} N\left[\phi(x;q)\right]\right\}_{q=0} \tag{4.17}$$

Here

$$N[\phi(x,q)] = D\phi(x;q) - {}^cD^{1/2}\phi(x,q) - \{\phi(x,q)\}^2 - 2x \tag{4.18}$$

Expanding in Taylor's Series w.r. to q,

$$\begin{aligned} \phi(x,q) &= \phi(x,0) + \sum_{m=1}^{\infty} \frac{1}{m!}\left(\frac{\partial^m \phi}{\partial q^m}\right)_{q=0} q^m. \\ &= \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{\partial^m \phi}{\partial q^m}\right)_{q=0} q^m \\ &= y_0(x) + \sum_{m=1}^{\infty} y_m q^m \quad \text{where } y_m = \frac{1}{m!}\left(\frac{\partial^m \phi}{\partial q^m}\right)_{q=0} \end{aligned} \tag{4.19}$$

When $q = 0$, $\phi(x;0) = y_0(x)$ and for $q = 1$,

$$\phi(x,1) = \sum_{i=1}^{\infty} y_i \tag{4.20}$$

Now,

$$\begin{aligned} \{\phi(x;q)\}^2 &= \left[\sum_{m=0}^{\infty} y_m q^m\right]^2 \\ &= \left(\sum_{m=0}^{\infty} y_m q^m\right)\left(\sum_{m=0}^{\infty} y_m q^m\right) \\ &= \sum_{m=1}^{\infty} \sum_{j=1}^{m-1} \left(y_{m-1-j}\, y_j\right) q^{m-1} \end{aligned} \tag{4.21}$$

Now,

$$\frac{\partial^{m-1}}{\partial q^{m-1}}\{\phi(x;q)\}^2_{q=0} = \sum_{j=0}^{m-1}(m-1)!y_{m-1-j}\,y_j$$

$$\Rightarrow \quad \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}\{\phi((x;q)\}^2_{q=0} = \sum_{j=0}^{m-1} y_{m-1-j}\,y_j \tag{4.22}$$

From (4.17), from $m = 1, 2, 3, \cdots$

$$\begin{aligned} &R_m[y_0, y_1, y_2, \cdots, y_{m-2}, y_{m-1}] \\ &\quad = \frac{1}{(m-1)!}\left\{\frac{\partial^{m-1}}{\partial q^{m-1}}N[\phi(x,q)]\right\}_{q=0} \\ &\quad = D\left[\frac{1}{(m-1)!}\left(\frac{\partial^{m-1}\phi}{\partial q^{m-1}}\right)_{q=0}\right] - {}^cD^{1/2}\left[\frac{1}{(m-1)!}\left(\frac{\partial^{m-1}\phi}{\partial q^{m-1}}\right)_{q=0}\right] \\ &\qquad - \left[\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}\{\phi(x;q)\}^2\right]_{q=0} - \frac{1}{(m-1)!}\left[\frac{\partial^{m-1}(2x)}{\partial q^{m-1}}\right]_{q=0} \end{aligned} \tag{4.23}$$

For $m \geq 1$:

$$\begin{aligned} &R_m[y_0, y_1, y_2, \cdots, y_{m-2}, y_{m-1}] \\ &\quad = Dy_{m-1} - {}^cD^{1/2}y_{m-1} - \sum_{j=0}^{m-1} y_{m-1-j}\,y_j - (1-\lambda_m)(2x) \end{aligned} \tag{4.24}$$

From (4.14): (Ham's iteration scheme)

$$\begin{aligned} &L[y_m(x) - \lambda_m y_{m-1}(x)] \\ &\quad = h\left[dy_{m-1} - {}^cD^{1/2}y_{m-1} - \sum_{j=0}^{m-1} y_{m-1-j}\,y_j - (1-\lambda_m)(2x)\right] \end{aligned} \tag{4.25}$$

We consider linear operator L as the highest order derivative of the differential equation (4.14) i.e., $L \equiv \frac{d}{dx}$ and $L^{-1} = I$.

Initial condition $y(0) = 1$

$$\Rightarrow \quad y_0(0) + y_1(0) + y_2(0) + \cdots = 1 \tag{4.26}$$

Assume $y_0(x) = 1$, then from (4.25), $y_0(0) = 1$ and $y_i(0) = 0$ for $i \geq 1$
From (4.25) for $m = 1$

$$L[y_1(x)] = h\left[Dy_0 - {}^cD^{1/2}y_0 - (y_0(x))^2 - 2x\right]$$

Integrating, $y_1(x) = -h(x + x^2)$
From (4.25), for $m \geq 2$

$$y_m = y_{m-1}(x) + h\left[IDy_{m-1}(x) - I^{1/2}y_{m-1}(x) - I\left(\sum_{j=0}^{m-1}(y_{m-j}\, y_j)\right)\right]$$

For $m = 2$,

$$y_2(x) = -h(1+h)(x+x^2) + h^2\left[\frac{4}{3\sqrt{\pi}}x^{3/2} + x^2 + \frac{16}{15\sqrt{\pi}}x^{5/2} + \frac{2}{3}x^3\right]$$

For $m = 3$,

$$\begin{aligned} y_3(x) = {} & (1+h)[\cdot\cdot\cdot\cdot\cdot] - 2h^3\left[\frac{8}{15\sqrt{\pi}}x^{5/2} + \frac{x^3}{3} + \frac{32}{105\sqrt{\pi}}x^{7/2} + \frac{x^4}{6}\right] \\ & -h^3\left(\frac{x^3}{3} + \frac{x^4}{2} + \frac{x^5}{5}\right) \end{aligned}$$

For $m = 4$,

$$\begin{aligned} y_4(x) = {} & (1+h)[\cdot\cdot\cdot\cdot\cdot] + h^4\left[\frac{8}{15\sqrt{\pi}}x^{5/2} + x^3 + \frac{336}{105\sqrt{\pi}}x^{7/2} + \frac{3}{2}x^4\right. \\ & \left. + \frac{48}{21\sqrt{\pi}}x^{9/2} + x^5 + \cdots\right] \end{aligned}$$

For $m = 5$,

$$\begin{aligned} y_5 = {} & \frac{x^3}{6} + \left(\frac{8}{\lceil 9/2}\right)x^{7/2} + \left(\frac{13}{8} + \frac{1}{4(\lceil 5/2\,)^2}\right)x^4 + \frac{247}{9\lceil 9/2}x^{9/2} \\ & + \left(\frac{503}{240} + \frac{8}{25(\lceil 5/2\,)^2}\right)x^5 + + \frac{3032}{1155\lceil 5/2}x^{11/2} + \cdots \end{aligned}$$

For $m = 6$,

$$y_6 = \frac{\lceil 4}{6\lceil 9/2}x^{7/2} + \frac{8}{\lceil 5}x^4 + \left[\left(\frac{13}{8} + \frac{1}{4(\lceil 5/2\,)^2} \right) \frac{\lceil 5}{\lceil 11/2} + \frac{16}{\lceil 9/2}\frac{2}{9} + \frac{4}{9}\cdot\frac{1}{\lceil 7/2} + \frac{2}{9}\cdot\frac{1}{\lceil 5/2} \right] x^{9/2} + \cdots$$

For $m = 7$,

$$y_7 = \frac{x^4}{24} + \frac{10}{\lceil 11/2}x^{9/2} + \cdots$$

For $h = -1$ and $\alpha = 1/2$,

$$\begin{aligned}
y_0(x) &= 1 \\
y_1(x) &= x + x^2 \\
y_2(x) &= \frac{4}{3\sqrt{\pi}}x^{3/2} + x^2 + \frac{16}{15\sqrt{\pi}}x^{5/2} + \frac{2x^3}{3} \\
y_3(x) &= \frac{x^2}{2} + \frac{32}{15\sqrt{\pi}}x^{5/2} + \frac{4x^3}{3} + \frac{128}{105\sqrt{\pi}}x^{7/2} + \frac{5}{6}x^4 + \frac{x^5}{5} \\
y_4(x) &= \frac{8}{15\sqrt{\pi}}x^{5/2} + x^3 + \frac{336}{105\sqrt{\pi}}x^{7/2} + \frac{3}{2}x^4 + \frac{48}{21\sqrt{\pi}}x^{9/2} + x^5 \\
y_5(x) &= \frac{x^3}{6} + \frac{128}{105\sqrt{\pi}}x^{7/2} + \left(\frac{13}{8} + \frac{1}{9\pi} \right) x^4 + \frac{247 \times 16}{9 \times 105\sqrt{\pi}}x^{9/2} + \cdots \\
y_6(x) &= \frac{80}{105\sqrt{\pi}}x^{7/2} + \frac{1}{3}x^4 + \cdots \\
y_7(x) &= \left(\frac{1}{8} + \frac{7}{9\pi} \right) x^4 + \cdots \\
y_{\text{HAM}} &= 1 + x + \frac{4}{3\sqrt{\pi}}x^{3/2} + \frac{5x^2}{2} + \frac{56}{15\sqrt{\pi}}x^{5/2} + \frac{19x^3}{6} + \frac{608}{105\sqrt{\pi}}x^{7/2} \\
&\quad + \left(\frac{53}{12} + \frac{4}{9\pi} \right) x^4 + \left(\frac{1256}{135\sqrt{\pi}} + \frac{1024}{2835\pi^{3/2}} \right) x^{9/2} + \cdots
\end{aligned}$$

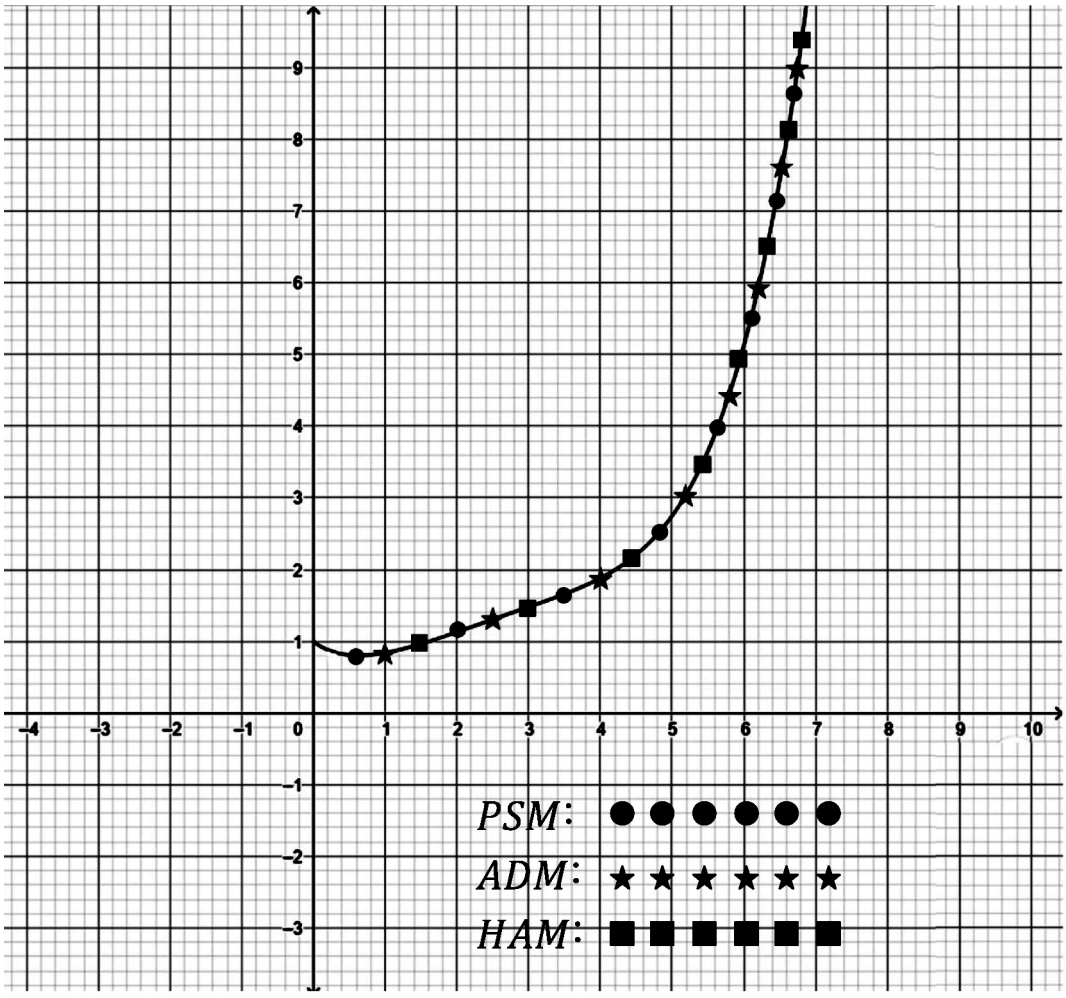

Fig. Graph for linear PDE

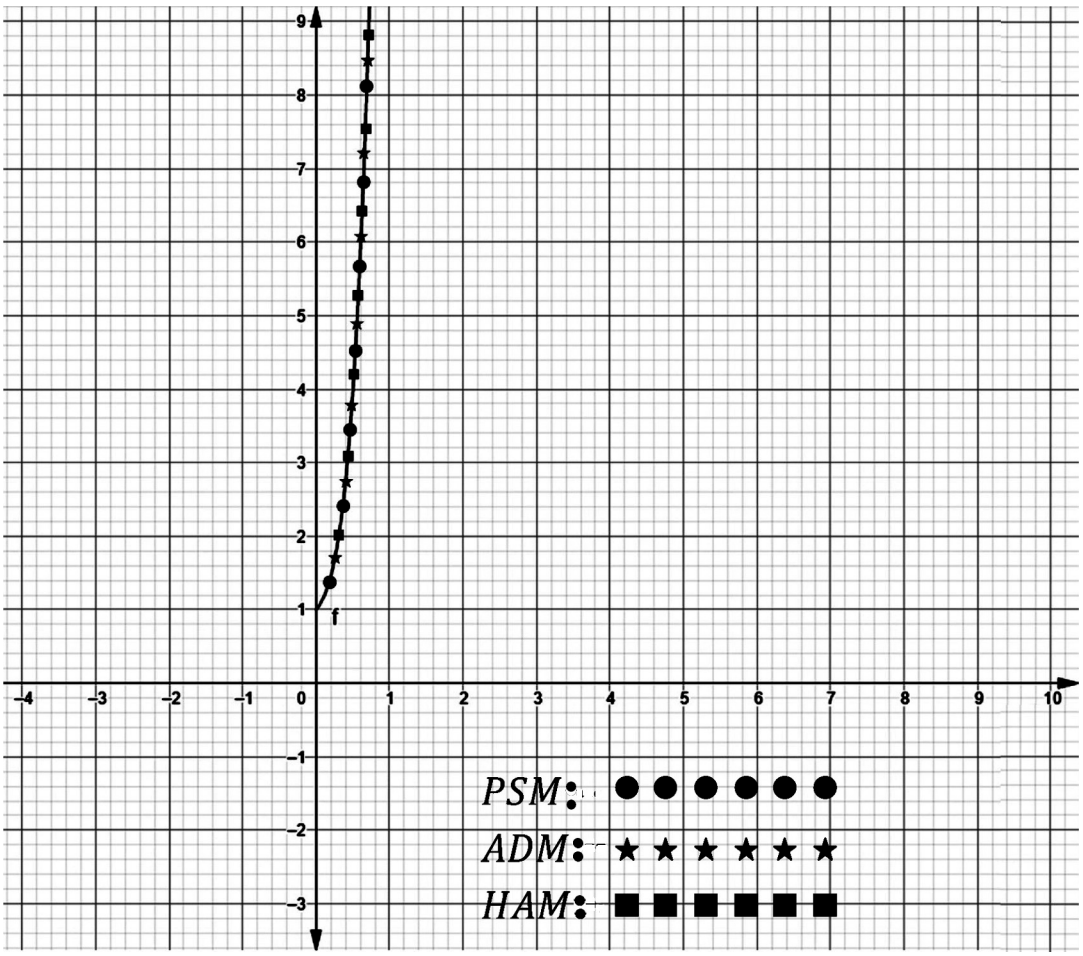

Fig. Graph for Non-linear PDE

Conclusion. For comparison of solutions obtained by HAM, ADM and PSM, the graphs of the solutions obtained by these methods both for linear and nonlinear PDE. It has been found that the graphs corresponding to the solutions upto certain number of terms of the above mentioned methods are exactly the same showing that each of the different methods has strong base for accuracy of the results, although the question of convergency may occur. It can be shown that convergence criterion can be proved in Banach space for these methods.

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